

Composite Structure and Causality

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Abstract We study the question of whether a composite structure of elementary particles, with a length scale $1/\Lambda$, can leave observable effects of non-locality and causality violation at higher energies (but $\lesssim \Lambda$). We formulate a model-independent approach based on Bogoliubov-Shirkov formulation of causality. We analyze the relation between the fundamental theory (of finer constituents) and the derived theory (of composite particles). We assume that the fundamental theory is causal and formulate a condition which must be fulfilled for the derived theory to be causal. We analyze the condition and exhibit possibilities which fulfil and which violate the condition. We make comments on how causality violating amplitudes can arise.

Keywords Causality · Composite structure · S-matrix methods

1 Introduction

The standard model (SM), a local quantum field theory, has served so far as a very good description of elementary particle processes [1]. It is however widely believed that soon, when higher energies are experimentally accessible, new phenomena may emerge that require a description that goes beyond the standard model. Among the various the possibilities, is the possibility that a composite nature of the standard model constituents may be revealed [2] and a possible failure of locality [3, 4]. It is possible that the underlying physics is nonlocal at shorter distances which could be a result of composite structure of particles, or granularity of space-time, or underlying noncommutative structure of space-time [5]. With a nonlocal interaction, often goes causality violation that can arise because the interaction region, encloses points separated by a space-like interval. Causality violation has been studied in the context of non-local [6, 7] and non-commutative quantum field theories [8–10]. It has in fact been suggested [6, 7, 11, 12] that non-local quantum field theories [13, 14] may indeed serve as effective field theories for a deeper/more fundamental theory such as a composite

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model; and the former indeed show causality violation [6, 7, 14]. An effective tool to study causality has been developed by Bogoliubov and Shirkov [15] and has been in particular employed for the causality violation in non-local [6, 7] and non-commutative QFT's [10]. We wish to consider the following question: in view of the possible composite nature of elementary particles, leading to extended structures, will these leave an observable effect in the form of a violation of causality and locality that can be detected? A similar question regarding a violation of the Pauli exclusion principle on account of the compositeness of particles has been earlier addressed to [16–18]. This question is particularly interesting since should there be a signal of CV, it will be detected long before an explicit knowledge of composite structure is known. In fact, it has been suggested [6, 7] that the unknown physics at high energy scales (Λ) from a possible source can effectively be represented in a consistent way (a unitary, gauge-invariant, finite (or renormalizable) theory) by a nonlocal theory at energies lower than Λ , but higher than the present ones. In other words, the nonlocal standard model, with a parameter Λ , can serve as such an effective field theory and will afford a model-independent way of consistently reparametrizing the effects beyond standard model. In this model, one finds that there is but a small CV at low energies, which grows rapidly as energies approach Λ and beyond these, the fundamental theory is expected to take over and presumably it leads to no CV again. The aim of the present work is to approach this question in a model-independent way in connection with a composite structure of SM constituents.

2 Preliminary

2.1 Definition of the Problem

Suppose that the presently known standard model particles are a composite of a set of finer constituents. Suppose that these underlying constituents belong to a local causality-preserving *fundamental* theory. Suppose, further that at lower energies, one only observes the composite bound states and their scattering processes. These bound state particles are extended objects. A priori, their interaction is expected to be non-local. A nonlocal covariant interaction has, at a given instant, interaction spread over a region in space, which therefore contains spatially separated points. An obvious question arises: will the interactions of the *composite* theory be such that causality is preserved by this low-energy theory? We need the fundamental theory for energy scales $\gg \Lambda$, and for energy scales $\ll \Lambda$, we have the set of composite particles described by the “derived” theory. Then the question, paraphrased differently is, will the *phase transition* (should there be one) from the fundamental to the composite be causality preserving or it could lead to a breakdown of causality at short enough distances?

2.2 Definition of the System

Let, for simplicity, the fundamental theory, denoted by \mathcal{F} , be characterized by a single coupling constant g . For the purpose of formulation of the Bogoliubov-Shirkov (BS) criterion of causality, we shall need to formulate a theory with a variable coupling $g(x)$. Let the low-energy derived theory be characterized by its own coupling constant $\lambda \equiv \lambda[g]$, which for identical reasons, we shall need to allow to depend on space-time: $\lambda = \lambda(x)$. We shall assume, for simplicity, that the derived theory, denoted by \mathcal{C} , is completely described by its scattering states: i.e. we shall assume that the model admits no bound states. A scattering state of the derived theory can be looked upon from *two* different point of views:

- a scattering state, as $t \rightarrow \pm\infty$, is a state of a certain set of non-interacting composite particles of the low energy theory with certain momenta, polarizations etc.
- a scattering state, as $t \rightarrow \pm\infty$, is a (complicated) configuration of fields of the fundamental theory.

3 Causality Formulation for a Theory without a Well-Defined S-Matrix

Bogoliubov and Shirkov have shown [15] that S -operator is causal *only if* it satisfies,

$$\frac{\delta}{\delta g(x)} \left[\frac{\delta S}{\delta g(y)} S^\dagger \right] = 0, \quad x \prec \sim y. \tag{1}$$

(Here, $x < y \iff x_0 < y_0$ and $x \sim y \iff (x - y)^2 < 0$). The condition is obtained from the primary meaning of causality that a disturbance does not propagate outside the forward light-cone (the disturbance considered is that in $g(x)$ ¹), and is independent of any specific field theory formulation. The BS causality criterion holds for a theory for which an S -operator is defined. For a theory such as QCD, some of the matrix elements of the S -operator may not exist on account of the infrared divergences. It is nonetheless true that an alternate formulation in terms of the U -operator (i.e. $U(-T, T')$) can be given. This is so because, the U -operator is unitary as much as the S -operator and the BS criterion depends on two points x, y with $x \prec \sim y$ which can always be chosen to be such that they both lie in $(-T, T')$. The relation would then read

$$\frac{\delta}{\delta g(x)} \left[\frac{\delta U}{\delta g(y)} U^\dagger \right] = 0, \quad x \prec \sim y; \quad -T < x_0, \quad y_0 < T'. \tag{2}$$

It is possible to alternately formulate the causality condition in terms of the following choices of the couplings. [This way results when we suitably integrate (2) over $x_0 < 0$ and $y_0 > 0$.] In this approach, we make a comparison of the following two *neighboring* theories in the coupling constant space:²

1. Fundamental theory \mathcal{F}' : Coupling constants = g'_2 (a constant value) for $x_0 > 0$ and g_1 (a constant value) for $x_0 < 0$. Corresponding derived theory is \mathcal{C}' .
2. Fundamental theory \mathcal{F}'' : Coupling constants = g''_2 (a constant value) for $x_0 > 0$ and g_1 (the same constant value) for $x_0 < 0$. Corresponding derived theory is \mathcal{C}'' .

All the coupling constants are (chosen to be) space-independent. It suffices for our purpose that g'_2 differs infinitesimally from g_1 and g''_2 . (We can in fact assume that the infinitesimal change from g_1 to g_2 is carried out adiabatically and in an infinitesimal time.) Then, we can alternately formulate³ the causality condition as,

$$U[g_1, g'_2; -T, T'] U^\dagger[g_1, g'_2; -T, T'] \text{ is independent of } g_1. \tag{3}$$

¹One may consider varying $g(x)$ an unphysical operation, but one can look alternately upon varying $g(x)$ at a point x_0 as insertion of a (specific) local operator $\frac{\partial \mathcal{L}_I}{\partial g(x)}$ at x_0 and study the propagation of its effects.

²The idea of varying the coupling with time over all space is not an entirely unfamiliar one: it is also employed in the LSZ formulation.

³For a simpler and intuitive understanding of the causality condition in either form, see e.g. S.D. Joglekar, hep-th/0601006.

This alternate formulation makes mathematics simpler, though it may lead to an unusual-looking Physics.

In the following, we shall adopt a “reductio ad absurdum” approach: We shall let, if possible, that the theory \mathcal{C} be causality-preserving and deduce the consequences of causality of \mathcal{F} for \mathcal{C} and analyze these.

4 Relations between the Derived Theory and the Fundamental Theory

4.1 Relations between Coupling Constants

The coupling constant λ is a function of g . If we allow a space-time dependent coupling, then $\lambda = \lambda[g]$. A small change⁴ $\delta g(x)$ in the coupling $g(x)$ about $g(x) = g = \text{constant}$, will cause a change in $\lambda(x)$ as given by⁵ $\delta\lambda(z) = \int dy \frac{\delta\lambda(z)}{\delta g(y)}|_{g(y)=g} \delta g(y)$. For the BS criterion of causality of (2), we need to know the impact of a *localized* change $g(x) \rightarrow g(x) + \delta g(x) \equiv g(x) + \varepsilon\delta^4(x - \tilde{x})$ on the function $\lambda(x)$. Now, if causality is valid, $\lambda(x)$ cannot be affected for any $x_0 < \tilde{x}_0$. Assuming that the theory has T -invariance,⁶ $\lambda(x)$ cannot be affected for any $x_0 > \tilde{x}_0$. Thus, this *together with causality* requires that,

$$\lambda(x) \rightarrow \lambda(x) + C\varepsilon\delta^4(x - \tilde{x}) + \text{terms having finite order derivatives of delta function.}$$

Thus,

$$\frac{\delta\lambda(z)}{\delta g(y)} = C\delta^4(z - y) + \text{terms having finite order derivatives of delta function.}$$

Then, for a *constant small* change $\delta g = \varepsilon$, for all $x_0 > 0$, [i.e. $\delta g(x) = \varepsilon\theta(x_0)$]; we find,

$$\begin{aligned} \delta\lambda(z) &= \int d^4y \frac{\delta\lambda(z)}{\delta g(y)} \delta g(y) \\ &= \int d^4y \{C\delta^4(z - y) + \text{derivatives of delta function}\} \varepsilon\theta(y_0) \\ &= C'\varepsilon\theta(z_0) \quad \text{for } z_0 > 0. \end{aligned}$$

We shall denote by $\lambda'_2 = \lambda[g'_2, g_1]$ and $\lambda''_2 = \lambda[g''_2, g_1]$.

4.2 Relation between States

We shall work in the interaction picture of \mathcal{C} . Let the derived theory \mathcal{C}' have as incoming states⁷ $\{|\tilde{c}_m(\lambda_1, -T)\rangle\}$ which, as $-T \rightarrow -\infty$, represents scattering states with a number of

⁴For the argument presented subsequently, we shall go back to a *general* space-time dependent coupling and not confine ourselves to the specific couplings presented in the previous section.

⁵We shall assume the existence and non-vanishing of $\frac{\delta\lambda(z)}{\delta g(y)}|_{g(y)=g}$. By translational invariance, this quantity is a function of $(z - y)$ and is independent of the point z as such.

⁶We shall need that the theory \mathcal{F} with a *variable* coupling has a T -invariance. This is possible to formulate a time-reversal transformation for a theory with a variable $g(x)$: we need to define the action of time-reversal as $Tg(\mathbf{x}, t)T^{-1} = g(\mathbf{x}, -t)$.

⁷For technical simplicity, we shall assume that the set of states is countably infinite.

free composite particles. We shall keep T finite and will let $T \rightarrow \infty$ only at the end of the argument. Evidently, as $-T \rightarrow -\infty$, $|\tilde{c}_m(\lambda_1, -T)\rangle$ depends on λ_1 only through the self-interaction of each individual non-interacting particle in the state. Let $\tilde{\mathcal{H}}$ denote the Hilbert space of states of \mathcal{C}' . Then the hypothesis that the scattering states of \mathcal{C}' forms a complete set implies that the set $\{|\tilde{c}_m(\lambda_1, -T)\rangle\}$ spans $\tilde{\mathcal{H}}$: $\tilde{\mathcal{H}} \equiv sp\{|\tilde{c}_m(\lambda_1, -T)\rangle\}$. We shall denote by \mathcal{H} , the Hilbert space of states of \mathcal{F}' (and likewise for \mathcal{F}''). Consider a state $|\tilde{c}_m(\lambda_1, -T)\rangle \in \tilde{\mathcal{H}}$ in the interaction picture. On physical grounds, we know that there is a corresponding state of \mathcal{F}' in the interaction picture, denoted by $|c_m(g_1, -T)\rangle$. We note that \mathcal{H} can, in addition, have states linearly independent of the states $\{|c_m(g_1, -T)\rangle\}$. We augment this set to complete an orthonormal basis $\{|c_m(g_1, -T)\rangle\} \cup \{|\beta_n(g_1, -T)\rangle\} \equiv \{|\alpha_p(g_1, -T)\rangle\}$ for \mathcal{H} . We shall call the span of $\{|c_m(g_1, -T)\rangle\}$ by $\hat{\mathcal{H}} \subset \mathcal{H}$. A similar discussion holds for \mathcal{F}'' . Let us now consider the time-evolution, from $t = -T$ to $t = T'$, of a single particle state of \mathcal{C}'' denoted by $|\tilde{s}_p(\lambda_1, -T)\rangle$, which belongs to the basis of $\tilde{\mathcal{H}}$. The unitary time evolution operator $\tilde{U}[\lambda_1, \lambda_2''; -T, T']$ as applied to the state leads to

$$\tilde{U}[\lambda_1, \lambda_2''; -T, T']|\tilde{s}_p(\lambda_1, -T)\rangle = |\tilde{s}_p(\lambda_2'', T')\rangle \in \tilde{\mathcal{H}}. \tag{4}$$

This state is a single particle state of slightly different mass, on account of a slightly different self-energy, and interacts with a coupling λ_2'' . We shall also introduce interaction picture states $|\tilde{d}_m(\lambda_2'', T')\rangle$. These states are at $t = T'$ and as $T' \rightarrow \infty$ consist of a set of non-interacting (but self-interacting) particles of a slightly different mass and coupling constant λ_2'' . These are analogues of the “out” states. We shall assume that these also span $\tilde{\mathcal{H}}$. We shall further make a convention: Under time reversal, the quantum numbers of particles in the state $|\tilde{c}_m(\lambda_1, -T)\rangle$ become those of $\langle \tilde{d}_m(\lambda_1, T') |$. Now, consider an exclusive process in \mathcal{C}'' . The magnitude of the quantum mechanical amplitude for it, as seen from \mathcal{C}'' and \mathcal{F}'' are identical, as these are, in principle, experimentally observable:

$$\begin{aligned} |\tilde{u}_{nm}| &\equiv |\langle \tilde{d}_n(\lambda_2'', T') | \tilde{U}[\lambda_1, \lambda_2''; -T, T'] | \tilde{c}_m(\lambda_1, -T) \rangle| \\ &\equiv |\langle d_n(g_2'', T') | U[g_1, g_2''; -T, T'] | c_m(g_1, -T) \rangle| \equiv |u_{nm}|. \end{aligned} \tag{5}$$

Here, we have introduced states $|d_n(g_2'', T')\rangle$ in \mathcal{H} analogous to $|\tilde{d}_n(\lambda_2'', T')\rangle$ in $\tilde{\mathcal{H}}$. We note that U here is the U -matrix in the interaction picture of \mathcal{F}'' , as the set of states $\{|c_m(g_1, -T)\rangle\}$ evolve according to the interaction Hamiltonian $\mathcal{H}'_I(g)$ (in the interaction picture) of the \mathcal{F}'' .

First we note that on account of unitarity of \tilde{U} and (5),

$$\begin{aligned} 1 &= \sum_n |\langle \tilde{d}_n(\lambda_2'', T') | \tilde{U}[\lambda_1, \lambda_2''; -T, T'] | \tilde{c}_m(\lambda_1, -T) \rangle|^2 \\ &= \sum_n |\langle d_n(g_2'', T') | U[g_1, g_2''; -T, T'] | c_m(g_1, -T) \rangle|^2 \end{aligned} \tag{6}$$

and

$$\begin{aligned} 1 &= \sum_m |\langle \tilde{d}_n(\lambda_2'', T') | \tilde{U}[\lambda_1, \lambda_2''; -T, T'] | \tilde{c}_m(\lambda_1, -T) \rangle|^2 \\ &= \sum_m |\langle d_n(g_2'', T') | U[g_1, g_2''; -T, T'] | c_m(g_1, -T) \rangle|^2. \end{aligned} \tag{7}$$

So, the unitarity of U implies,

$$\begin{aligned} \langle d_n(g_2'', T') | U [g_1, g_2''; -T, T'] | \beta_m(g_1, -T) \rangle &= 0, \\ \langle \beta_n(g_2'', T') | U [g_1, g_2''; -T, T'] | c_m(g_1, -T) \rangle &= 0. \end{aligned} \tag{8}$$

The relations (8) implies that U is a block-diagonal matrix. The unitarity of U then implies that the block corresponding to the subspace $\hat{\mathcal{H}}$, viz. \hat{U} , is also unitary. We shall now attempt relate these further. In this connection, we recall a result for a finite dimensional matrices:

Lemma *Let U and U' be two $N \times N$ unitary matrices satisfying: $|u'_{ij}| = |u_{ij}|$; $1 \leq i, j \leq N$. Then, there exist phases $\{\theta_i : i = 1, 2, \dots, N\}$ and $\{\phi_i : i = 2, \dots, N\}$ such that $u'_{ij} = u_{ij} \exp [i(\theta_i + \phi_j)] : 1 \leq i, j \leq N$ with $\phi_1 \equiv 0$.*

Proof Let the diagonal elements of U' and U be related by: $u'_{ii} = \exp(i\Theta_i)u_{ii}$. We define U'' by $u''_{ij} = \exp(-i\Theta_i)u'_{ij}$. Then, $u''_{ii} = u_{ii}$. Now U'' is unitary and thus, a priori, has N^2 independent parameters. The information on moduli of elements constitutes $(N - 1)^2$ independent conditions, corresponding to an $(N - 1) \times (N - 1)$ dimensional submatrix; the rest of the $(2N - 1)$ moduli being determined by relations implying that the norm of each row and column is unity. The relations $u''_{ii} = u_{ii}$ imply additional N relations on the phases on u_{ii} . This leaves $N^2 - (N - 1)^2 - N = N - 1$ free parameters. The phases of u''_{1j} , $2 \leq j \leq N$ are unconstrained by $|u''_{ij}| = |u_{ij}| : 1 \leq i, j \leq N$ and $u''_{ii} = u_{ii}$ and we define $u''_{1j} = u_{1j} \exp(i\phi_j)$, $2 \leq j \leq N$. Then, there are no free parameters and must lead to a unique U'' . Now, U'' specified by $u''_{ij} = u_{ij} \exp(i\phi_j - i\phi_i)$, $1 \leq i, j \leq N$ ($\phi_1 \equiv 0$) is such a solution. This together with $u''_{ij} = \exp(-i\Theta_i)u'_{ij}$ leads to the result; with the definition $\Theta_i - \phi_i = \theta_i$.

Thus, in view of the unitarity of \tilde{U} , and \hat{U} and (5), we write,

$$\begin{aligned} \langle \tilde{d}_n(\lambda_2'', T') | \tilde{U} [\lambda_1, \lambda_2''; -T, T'] | \tilde{c}_m(\lambda_1, -T) \rangle \\ \equiv \langle d_n(g_2'', T') | U [g_1, g_2''; -T, T'] | c_m(g_1, -T) \rangle \times \exp(i\theta_n'' + i\phi_m). \end{aligned} \tag{9}$$

We shall assume that \mathcal{F} and \mathcal{C} are have time-reversal invariance and derive the consequences. Under time reversal, we know then that,

$$\langle \beta | S | \alpha \rangle = \langle T \alpha | S | T \beta \rangle \tag{10}$$

where $|T\beta\rangle$ is the state obtained by time-reversing the quantum numbers of the state $|\beta\rangle$. In this case, it would imply, keeping in mind our choice of definitions,

$$\begin{aligned} \langle \tilde{d}_n(\lambda_2'', T') | \tilde{U} [\lambda_1, \lambda_2''; -T, T'] | \tilde{c}_m(\lambda_1, -T) \rangle \\ = \langle \tilde{d}_m(\lambda_1, T) | \tilde{U} [\lambda_1, \lambda_2''; -T, T'] | \tilde{c}_n(\lambda_2'', -T') \rangle. \end{aligned} \tag{11}$$

We write a similar relation for \mathcal{F} . Putting $T' = T$ (or equivalently, noting that the matrix elements are insensitive to T' and T), we find,

$$\phi_p(\lambda_2, \lambda_1) = \theta_p(\lambda_1, \lambda_2). \tag{12}$$

□

5 Consequence of Causality of \mathcal{F} for \mathcal{C}

We shall assume that the fundamental theory \mathcal{F} is causal and deduce the consequences for the derived theory $\mathcal{C}(\mathcal{C}', \mathcal{C}'')$. The causality of \mathcal{F} implies that

$$U[g_1, g_2''; -T, T']U^\dagger[g_1, g_2'; -T, T']$$

is independent of g_1 . Hence,

$$\mathcal{M}_{nm} \equiv \langle d_n(g_2'', T') | U[g_1, g_2''; -T, T'] U^\dagger[g_1, g_2'; -T, T'] | d_m(g_2', T') \rangle$$

is also independent of g_1 since the state vectors $\langle d_n(g_2'', T') |$ and $| d_m(g_2', T') \rangle$ are independent of g_1 with g_2' and g_2'' fixed. We shall re-express \mathcal{M}_{nm} in terms of the matrix elements of the derived theory $\mathcal{C}(\mathcal{C}', \mathcal{C}'')$ and deduce the consequences. We note,

$$\begin{aligned} \mathcal{M}_{nm} &= \langle d_n(g_2'', T') | U[g_1, g_2''; -T, T'] U^\dagger[g_1, g_2'; -T, T'] | d_m(g_2', T') \rangle \\ &= \sum_p \langle d_n(g_2'', T') | U[g_1, g_2''; -T, T'] | \alpha_p(g_1, -T) \rangle \\ &\quad \times \langle \alpha_p(g_1, -T) | U^\dagger[g_1, g_2'; -T, T'] | d_m(g_2', T') \rangle \end{aligned} \tag{13}$$

$$\begin{aligned} &= \sum_p \langle d_n(g_2'', T') | U[g_1, g_2''; -T, T'] | c_p(g_1, -T) \rangle \\ &\quad \times \langle c_p(g_1, -T) | U^\dagger[g_1, g_2'; -T, T'] | d_m(g_2', T') \rangle \end{aligned} \tag{14}$$

$$\begin{aligned} &= \sum_p \langle \tilde{d}_n(\lambda_2'', T') | \tilde{U}[\lambda_1, \lambda_2''; -T, T'] | \tilde{c}_p(\lambda_1, -T) \rangle \exp[-i(\tilde{\theta}_p'' - \tilde{\theta}_p')] \\ &\quad \times \langle \tilde{c}_p(\lambda_1, -T) | \tilde{U}^\dagger[\lambda_1, \lambda_2'; -T, T'] | \tilde{d}_m(\lambda_2', T') \rangle \exp[-i(\theta_p'' - \theta_p')] \end{aligned} \tag{15}$$

$$\equiv \tilde{\mathcal{M}}_{nm}(\lambda_2'', \lambda_2', \lambda_1). \tag{16}$$

In the 3rd step, we have employed (8) and in the second step, we have employed the closure relation for \mathcal{F} .

In the above, $\theta_p'' \equiv \theta_n(\lambda_2'', \lambda_1)$, $\theta_p' \equiv \theta_m(\lambda_2', \lambda_1)$, and $\tilde{\theta}_p'' \equiv \theta_p(\lambda_1, \lambda_2'')$ etc. Thus, the expression (15) is independent of λ_1 :

$$\frac{\partial \tilde{\mathcal{M}}_{nm}(\lambda_2'', \lambda_2', \lambda_1)}{\partial \lambda_1} = 0. \tag{17}$$

6 Analysis of Causality Condition

We shall now analyze the condition (17) obtained as an implication of causality of \mathcal{F} . For this purpose, we shall find it useful to Taylor-expand θ_n as follows:⁸

$$\theta_n(\lambda_2'', \lambda_1) = \theta_n(\lambda_2', \lambda_{1(0)}) + \beta_n \Delta_1 + \gamma_n \Delta_2 + \delta_n \Delta_1 \Delta_2 + \dots$$

⁸Throughout, we have employed only the infinitesimal variations in the couplings. These are sufficient to determine the first order partial derivatives with respect to each λ_1 and λ_2 . Hence, we shall content ourselves with expansion only upto $O(\Delta_1 \Delta_2)$.

$$\begin{aligned} &\equiv \alpha_n + \beta_n \Delta_1 + \gamma_n \Delta_2 + \delta_n \Delta_1 \Delta_2 + \dots, \\ \theta_m(\lambda'_2, \lambda_1) &= \alpha_m + \beta_m \Delta_1 + \dots. \end{aligned} \tag{18}$$

Here, $\Delta_1 \equiv \lambda_1 - \lambda_{1(0)}$; $\Delta_2 \equiv \lambda'_2 - \lambda'_2$; β, γ, δ refer to appropriate partial derivatives at $(\lambda'_2, \lambda_{1(0)})$ and $\lambda_{1(0)}$ is some value near λ_1 .

We note that if

$$\theta_n(\lambda_2, \lambda_1) \text{ is a function only of its first argument} \tag{I}$$

then, $(\tilde{\theta}''_p - \tilde{\theta}'_p) \equiv \theta_p(\lambda_1, \lambda''_2) - \theta_p(\lambda_1, \lambda'_2)$ is zero and $(\theta''_n - \theta'_m) \equiv \theta_n(\lambda''_2, \lambda_1) - \theta_m(\lambda'_2, \lambda_1)$ is independent of λ_1 . Also, we can then carry out the sum over p using the completeness relation and find that the independence from λ_1 of

$$\begin{aligned} &\tilde{\mathcal{M}}_{nm}(\lambda''_2, \lambda'_2, \lambda_1) \\ &= \langle \tilde{d}_n(\lambda''_2, T') | \tilde{U}[\lambda_1, \lambda''_2; -T, T'] \tilde{U}^\dagger[\lambda_1, \lambda'_2; -T, T'] | \tilde{d}_m(\lambda'_2, T') \rangle \exp[-i(\theta''_n - \theta'_m)] \end{aligned} \tag{19}$$

for all m, n implies $\tilde{U}[\lambda_1, \lambda''_2; -T, T'] \tilde{U}^\dagger[\lambda_1, \lambda'_2; -T, T']$ is independent of λ_1 . This condition is indeed necessary for causality of \mathcal{C} . In fact, in *this* case, we can rewrite⁹

$$\begin{aligned} &\langle \tilde{d}_n(\lambda_2, T') | \tilde{U}[\lambda_1, \lambda_2; -T, T'] | \tilde{c}_m(\lambda_1, -T) \rangle \\ &\equiv \langle d_n(g_2, T') | U[g_1, g_2; -T, T'] | c_m(g_1, -T) \rangle \exp(i\theta_n(\lambda_2, \lambda_1) + i\theta_m(\lambda_1, \lambda_2)) \end{aligned} \tag{20}$$

as,

$$\begin{aligned} &\langle \tilde{d}^*_n(\lambda_2, T') | \tilde{U}[\lambda_1, \lambda_2; -T, T'] | \tilde{c}^*_m(\lambda_1, -T) \rangle \\ &\equiv \langle d_n(g_2, T') | U[g_1, g_2; -T, T'] | c_m(g_1, -T) \rangle \end{aligned} \tag{21}$$

by redefining states by absorbing phases:

$$(|\tilde{c}^*_m(\lambda_1, -T)\rangle = e^{-i\theta_m(\lambda_1)} |c_m(\lambda_1, -T)\rangle)$$

etc. We note that this redefinition of the states is meaningful and *compatible with causality* when θ_n is independent of its second argument. If on the other hand, θ_n is dependent on its second argument (excepting a possibility below), we cannot absorb a phase in a manner *compatible with causality*: a state $|\tilde{c}^*_m\rangle$ at $t = -T$ cannot be made to depend on the value of coupling λ_2 it would have at a later time $t > 0$.

We can, in fact, liberalize somewhat the above condition by requiring that,

$$\beta_n = \beta \quad \text{and} \quad \delta_n = 0, \quad \forall n. \tag{II}$$

In this case,

$$\begin{aligned} &(\tilde{\theta}''_p - \tilde{\theta}'_p) \equiv \theta_p(\lambda_1, \lambda''_2) - \theta_p(\lambda_1, \lambda'_2) \\ &= \beta \Delta_2 + \dots \end{aligned} \tag{22}$$

⁹We have dropped primes on λ_2 .

is independent of λ_1 and does not depend also on p and thus comes out of the summation in (15). The summation in (15) can be carried out using the completeness relation. Also, $(\theta''_n - \theta'_m) \equiv \theta_n(\lambda'_2, \lambda_1) - \theta_m(\lambda'_2, \lambda_1)$ is still independent of λ_1 . Thus, the entire discussion proceeds as before: in particular, as a little analysis shows, the phases can again be absorbed into the definition of states in a manner compatible with causality.

While we shall not provide the general analysis of (17), we shall establish examples of a few specific *sufficient* conditions for causality violation. (These are simple conditions that, in fact, contradict (I) or (II) above.) We can easily verify the following results:

1. There is causality violation if (i) for some n , $\delta_n \neq 0$ and (ii) $\beta_n = \beta_m, \forall n, m$.
2. There is causality violation if there be $m \neq n$ such that $\mathcal{M}_{nm}(\lambda''_2, \lambda'_2, \lambda_1) \neq 0$, when evaluated to $O(\Delta)$, and $\beta_m \neq \beta_n$.

Proof We shall let, if possible, \mathcal{C} be causal. We can then write,

$$\tilde{U}[\lambda_1, \lambda'_2; -T, T'] = \tilde{U}[\lambda''_2, 0, T']\tilde{U}[\lambda_1; -T, 0]. \tag{23}$$

Then, we can write the expression (15) as,

$$\begin{aligned} \tilde{\mathcal{M}}_{nm}(\lambda''_2, \lambda'_2, \lambda_1) &= \sum_p \langle \tilde{d}_n(\lambda''_2, 0) | \tilde{c}_p(\lambda_1, 0) \rangle \exp[-i(\tilde{\theta}''_p - \tilde{\theta}'_p)] \\ &\quad \times \langle \tilde{c}_p(\lambda_1, 0) | \tilde{d}_m(\lambda'_2, 0) \rangle \exp[-i(\theta''_n - \theta'_m)] \\ &\equiv \langle \tilde{d}_n(\lambda''_2, 0) | \mathcal{X} | \tilde{d}_m(\lambda'_2, 0) \rangle \exp[-i(\theta''_n - \theta'_m)] \end{aligned} \tag{24}$$

where $\mathcal{X} \equiv \sum_p | \tilde{c}_p(\lambda_1, 0) \rangle \langle \tilde{c}_p(\lambda_1, 0) | \exp[-i(\tilde{\theta}''_p - \tilde{\theta}'_p)]$. We shall now expand the quantities involved to the first order in the infinitesimals as in (18). In addition, we note that to the zeroth order in Δ_2 (i.e. $\lambda''_2 - \lambda'_2 = 0$), we have, $(\tilde{\theta}''_p - \tilde{\theta}'_p) = 0$ and the completeness relation leads to $\mathcal{X} = 1$. We further define,

$$\langle \tilde{d}_n(\lambda''_2, 0) | \tilde{d}_m(\lambda'_2, 0) \rangle = \delta_{nm} + i\eta_{nm}\Delta_2 + \dots \tag{25}$$

□

Proof of (i) We define $\delta_0 \equiv \max\{|\delta_n|\}$; and let $\pm\delta_0 = \delta_q$ for some q . We now have,

$$\begin{aligned} (\tilde{\theta}''_p - \tilde{\theta}'_p) &\equiv \theta_p(\lambda_1, \lambda''_2) - \theta_p(\lambda_1, \lambda'_2) \\ &= \beta\Delta_2 + \delta_p\Delta_1\Delta_2 + \dots \end{aligned} \tag{26}$$

and thus,

$$\begin{aligned} \mathcal{X} &\equiv \sum_p | \tilde{c}_p(\lambda_1, 0) \rangle \langle \tilde{c}_p(\lambda_1, 0) | \exp[-i(\tilde{\theta}''_p - \tilde{\theta}'_p)] \\ &= \exp(-i\beta\Delta_2) \sum_p | \tilde{c}_p(\lambda_1, 0) \rangle \langle \tilde{c}_p(\lambda_1, 0) | \exp[-i(\delta_p\Delta_1\Delta_2)] \\ &= \exp(-i\beta\Delta_2) \left[\mathcal{I} - i\Delta_1\Delta_2 \sum_p | \tilde{c}_p(\lambda_1, 0) \rangle \langle \tilde{c}_p(\lambda_1, 0) | \delta_p \right]. \end{aligned} \tag{27}$$

Thus,

$$\begin{aligned} & \exp(i\beta\Delta_2)\langle\tilde{d}_q(\lambda_2'',0)|\mathcal{X}|\tilde{d}_q(\lambda_2',0)\rangle \\ & = 1 + i\eta_{qq}\Delta_2 - i\Delta_1\Delta_2\sum_p\delta_p|u_{pq}|^2 + \dots \end{aligned} \tag{28}$$

where $u_{pq} \equiv \langle\tilde{d}_q(\lambda_2,0)|\tilde{c}_p(\lambda_1,0)\rangle$ (we can ignore primes on λ_2 in this term). The multiplicative exponential factor in (24) becomes:

$$\exp(-i\gamma_q\Delta_2 - i\delta_q\Delta_1\Delta_2 + \dots) \approx 1 - i\gamma_q\Delta_2 - i\delta_q\Delta_1\Delta_2 + \dots$$

Thus,

$$\begin{aligned} \tilde{\mathcal{M}}_{qq} & = 1 + i\eta_{qq}\Delta_2 - i\gamma_q\Delta_2 - i\delta_q\Delta_1\Delta_2 - i\Delta_1\Delta_2\sum_p\delta_p|u_{pq}|^2 + \dots \\ & = 1 + i\eta_{qq}\Delta_2 - i\gamma_q\Delta_2 - i\Delta_1\Delta_2\sum_p[\delta_q + \delta_p]|u_{pq}|^2 + \dots \end{aligned} \tag{29}$$

In view of the fact that either $\delta_p + \delta_q \geq 0, \forall p$ or $\delta_p + \delta_q \leq 0, \forall p$ the last term is necessarily non-vanishing and dependent on Δ_1 .¹⁰ □

Proof of (ii) Consider the matrix element

$$\tilde{\mathcal{M}}_{nm}(\lambda_2'',\lambda_2',\lambda_1) \equiv \langle\tilde{d}_n(\lambda_2'',0)|\mathcal{X}|\tilde{d}_m(\lambda_2',0)\rangle \exp[-i(\theta_n'' - \theta_m')] \neq 0 \tag{30}$$

for $n \neq m$. To the first order in the infinitesimals, the nonzero matrix element

$$\langle\tilde{d}_n(\lambda_2'',0)|\mathcal{X}|\tilde{d}_m(\lambda_2',0)\rangle$$

is independent of Δ_1 . The multiplicative phase factor,

$$\exp[-i(\theta_n'' - \theta_m')] = \exp\{-i(\alpha_n - \alpha_m) - i(\beta_n - \beta_m)\Delta_1 - i\gamma_n\Delta_2\}$$

is necessarily dependent on Δ_1 , thus implying causality violation. □

7 Additional Comments

We comment in a qualitative way upon how a phase factor depending on both values of the coupling can arise. Suppose that the derived theory \mathcal{C} is actually correctly described by a nonlocal covariant theory with a finite non-zero non-locality scale $\Delta \sim 1/\Lambda$. Since the theory is covariant, it is also non-local in time. We write,

$$\tilde{U}(\lambda_1,\lambda_2;-T,T') = \tilde{U}(\lambda_2;\Delta,T')\tilde{U}(\lambda_1,\lambda_2;-\Delta,\Delta)\tilde{U}(\lambda_1;-T,-\Delta) \tag{31}$$

where the first and the third factors on the right hand side depends only on one value of the coupling due to finite size of non-locality in time. The second factor however depends

¹⁰There is the obvious exception that $\delta_p = -\delta_q$ for every such p such that $u_{pq} \neq 0$; and this has to be valid for each such q for which $\delta_q = \pm\delta_0$.

on both couplings because in this time-interval $(-\Delta, \Delta)$, time evolution depends on both values of the coupling λ . On the other hand, the fundamental theory, being local and causal, however has no such analogue. The matrix $\tilde{U}(\lambda_1, \lambda_2; -\Delta, \Delta)$ can then give rise to phases depending on both couplings in relation (9).

Naively, one may expect that if the fundamental theory is causal, the derived theory should be so. Examples are however known where the diagrams of the fundamental theory are associated with a different weight in the actual phenomenology. For example, OZI rule in hadronic phenomenology gives a suppression of a subset of the QCD diagrams. While such a possibility is distinct from what is discussed in this work, generally such a modification of the amplitudes within the fundamental theory may alter the underlying properties of the fundamental theory such as causality.

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